

## RELATIONS IN A LOOP SOLITON AS A QUANTIZED ELASTICA

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ABSTRACT. In the previous article (J. Geom. Phys. **43** (2002) 146), we show the hyperelliptic solutions of a loop soliton as a study of a quantized elastica. This article gives some functional relations in a loop soliton as a quantized elastica.

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## 1. INTRODUCTION

We proposed problems of the quantized elastica in [Ma6]. Let a circle  $S^1$  immersed in a complex plane  $\mathbb{C}$  characterized by the affine coordinate  $Z(s) := X^1(s) + \sqrt{-1}X^2(s)$  around the origin. Here  $s$  is a parameter of  $S^1$  satisfying  $ds^2 = (dX^1)^2 + (dX^2)^2$ . For loops with the Euler-Bernoulli energy functional,

$$(1.1) \quad \mathcal{E}[Z] = \oint ds \{Z, s\}_{\text{SD}},$$

where  $\{Z, s\}_{\text{SD}}$  is the Schwarz derivative,

$$(1.2) \quad \{Z, s\}_{\text{SD}} := \partial_s \left( \frac{\partial_s^2 Z}{\partial_s Z} \right) - \frac{1}{2} \left( \frac{\partial_s^2 Z}{\partial_s Z} \right)^2,$$

the quantized elastica problem is to compute the “partition function”,

$$(1.3) \quad \mathcal{Z}[\beta] = \int DZ \exp(-\beta \mathcal{E}[Z]),$$

in a certain physical sense. Here  $\partial_s := \partial/\partial s$ . Whereas the ordinary (classical) elastica problem is to compute its extremal points of the energy functional (1.1), in the quantized elastica problem, we should calculate some contributions from loops with outside of the extremal points of (1.1), which contrasts with the classical elastica problem.

In order to make this physical functional (1.3) have mathematical meanings, we should classify a loop space  $\Omega\mathbb{C}$  of the complex plane with paying attentions upon the energy functional (1.1) and euclidean moves. In [Ma6], we studied the loop space as the moduli space of the quantized elastica,

$$\mathcal{M}_{\text{elas}}^{\mathbb{C}} := \{Z : S^1 \rightarrow \mathbb{C} \mid \oint dZ = 2\pi\} / \sim,$$

where  $\sim$  means the euclidean moves.  $\mathcal{M}_{\text{elas}}^{\mathbb{C}}$  has a spectrum decomposition,

$$(1.4) \quad \mathcal{M}_{\text{elas}}^{\mathbb{C}} := \prod_E \mathcal{M}_{\text{elas},E}^{\mathbb{C}}, \quad \mathcal{M}_{\text{elas},E}^{\mathbb{C}} := \{Z \in \mathcal{M}_{\text{elas}}^{\mathbb{C}} \mid \mathcal{E}[Z] = E\}.$$

As the loop soliton, which is defined as follows, preserves the local length and the energy functional (1.1), [Ma1, Ma6] shows that  $\mathcal{M}_{\text{elas},E}^{\mathbb{C}}$  consists of the orbits of a group of its related loop soliton.

**Definition 1.1.** *A one parameter family of a loop  $\{Z(t) \mid t \in \mathbb{R}\}$  for a real parameter  $t \in \mathbb{R}$  is called a loop soliton, if its half curvature  $q := \frac{1}{2\sqrt{-1}} \partial_s \log \partial_s Z(t, s)$  obeys the modified Korteweg-de Vries (MKdV) equation,*

$$\partial_t q + 6q^2 \partial_s q + \partial_s^3 q = 0,$$

where  $\partial_t := \partial/\partial t$ .

Using the loop soliton, we classified the loop space  $\mathcal{M}_{\text{elas}}^{\mathbb{C}}$  in the category of differential geometry and investigated its topological properties in [Ma6]. As a loop soliton is expressed by a hyperelliptic function of genus  $g$  if we include the infinite genus  $g = \infty$ , we have an expression of the partition functions as follows.

$$(1.5) \quad \mathcal{Z}[\beta] = \sum_{g=0}^{\infty} \mathcal{Z}^{(g)}[\beta], \quad \mathcal{Z}^{(g)}[\beta] := \int_{\mathcal{M}_{\text{elas},g}^{\mathbb{C}}} DZ \exp(-\beta \mathcal{E}[Z]),$$

where  $\mathcal{M}_{\text{elas},g}^{\mathbb{C}}$  is a subspace of  $\mathcal{M}_{\text{elas}}^{\mathbb{C}}$ , whose elements are expressed by hyperelliptic functions of genus  $g$ . We give explicit function forms of loop soliton in terms of Weierstrass hyperelliptic functions in [Ma3].

As J. McKay pointed out, there are apparent resemblances between relations in the replicable functions [FMN, Mc, MS] and those in the quantized elastica. Thus we have progressed investigations of the resemblances. In this article, we will study functional properties of the loop solitons and a quantized elastica as a sequel of the previous paper [Ma3]. Our investigations are closely related to the geometry of MKdV equation studied by Previato in [P] as mentioned in Remark 3.1 and 3.2.

Section 2 gives minimal preliminaries to express our results in §3 and §4. We start with a hyperelliptic curve given by (2.1) and thus there principally appear no other parameters beside  $\lambda$ 's in (2.1). The quantities defined in Definition 2.3 directly play important roles in our theory. After reviewing the previous results [Ma3] in Proposition 3.1, we give our main theorem in Theorem 3.1. (3.6) is a differential expression as the generalization to a general genus of (4.6) in [Ma3] for the genus one case,

$$Z^{(a)}(u) = \lim_{\epsilon \rightarrow 0} \int_{\sigma(\epsilon)}^u du \frac{1}{\sigma(\epsilon)^2} \exp \left( -\frac{1}{2} \int_{\epsilon}^u \int_0^{u'} \left[ \{Z^{(a)}(u''), u''\}_{\text{SD}} \right. \right. \\ \left. \left. - \{Z^{(a)}(u'' - \omega_a), u''\}_{\text{SD}} \right] du'' du' \right),$$

which was found under the stimulus of a formula obtained by J. McKay. Further we will investigate the properties of a quantized elastica based upon the fourier analysis in §4 and give our other main results in Proposition 4.1 and 4.2.

## 2. PRELIMINARY FOR HYPERELLIPTIC FUNCTIONS

*Hyperelliptic Curve:* This article deals with a hyperelliptic curve  $C_g$  of genus  $g$  ( $g > 0$ ) given by the affine equation,

$$(2.1) \quad \begin{aligned} y^2 &= f(x) \\ &= \lambda_{2g+1}x^{2g+1} + \lambda_{2g}x^{2g} + \cdots + \lambda_2x^2 + \lambda_1x + \lambda_0 \\ &= (x - b_1)(x - b_2) \cdots (x - b_{2g+1}), \end{aligned}$$

where  $\lambda_{2g+1} \equiv 1$  and  $\lambda_j$ 's and  $b_j$ 's ( $b_i = a_i, b_{g+i} = c_i$ ) are complex numbers.

**Definition 2.1.** [Ba1, Ba2, BEL1, W1]

(1) For a point  $(x_i, y_i) \in C_g$ , the unnormalized differentials of the first kind are defined by,

$$du_1^{(i)} := \frac{dx_i}{2y_i}, \quad du_2^{(i)} := \frac{x_i dx_i}{2y_i}, \quad \dots, \quad du_g^{(i)} := \frac{x_i^{g-1} dx_i}{2y_i}.$$

(2) The Abel map from  $g$ -th symmetric product of the curve  $C_g$  to  $\mathbb{C}^g$  is defined by,

$$u := (u_1, \dots, u_g) : \text{Sym}^g(C_g) \longrightarrow \mathbb{C}^g,$$

$$\left( u_k((x_1, y_1), \dots, (x_g, y_g)) \right) := \sum_{i=1}^g \int_{\infty}^{(x_i, y_i)} du_k^{(i)}.$$

**Notation 2.1.** Let us denote the homology of a hyperelliptic curve  $C_g$  by  $H_1(C_g, \mathbb{Z}) = \bigoplus_{j=1}^g \mathbb{Z}\alpha_j \oplus \bigoplus_{j=1}^g \mathbb{Z}\beta_j$ . Here these intersections are given as  $[\alpha_i, \alpha_j] = 0$ ,  $[\beta_i, \beta_j] = 0$  and  $[\alpha_i, \beta_j] = \delta_{i,j}$ . The complete hyperelliptic integral of the first kind are defined by,

$$\omega' := \frac{1}{2} \left[ \left( \int_{\alpha_j} du_i^{(a)} \right)_{ij} \right], \quad \omega'' := \frac{1}{2} \left[ \left( \int_{\beta_j} du_i^{(a)} \right)_{ij} \right], \quad \omega := \begin{bmatrix} \omega' \\ \omega'' \end{bmatrix}.$$

The Jacobi varieties (Jacobian)  $\mathcal{J}_g$  is defined as a complex torus,

$$\mathcal{J}_g := \mathbb{C}^g / \Lambda_g.$$

Here  $\Lambda_g$  is a real  $2g$ -dimensional lattice generated by the periodic matrix given by  $2\omega$ . Further  $u$  is the coordinate of  $\mathbb{C}^g$  and of the Jacobian  $\mathcal{J}_g$ .

**Definition 2.2.** Using the unnormalized differentials of the second kind,

$$dr_j^{(i)} = \frac{1}{2y_i} \sum_{k=j}^{2g-j} (k+1-j) \lambda_{k+1+j} x_i^k dx_i, \quad (j = 1, \dots, g),$$

the complete hyperelliptic integral matrices of the second kind are defined by,

$$\boldsymbol{\eta}' := \frac{1}{2} \left[ \left( \int_{\alpha_j} dr_i^{(a)} \right)_{ij} \right], \quad \boldsymbol{\eta}'' := \frac{1}{2} \left[ \left( \int_{\beta_j} dr_i^{(a)} \right)_{ij} \right].$$

The hyperelliptic  $\sigma$  function, which is a holomorphic function over  $u \in \mathbb{C}^g$ , is defined by [[Ba2], p.336, p.350], [Kl1, BEL1],

$$(2.2) \quad \sigma(u) := \sigma(u; C_g) := \gamma \exp\left(-\frac{1}{2} {}^t u \boldsymbol{\eta}' \boldsymbol{\omega}'^{-1} u\right) \vartheta \left[ \begin{smallmatrix} \delta'' \\ \delta' \end{smallmatrix} \right] \left( \frac{1}{2} \boldsymbol{\omega}'^{-1} u; \boldsymbol{\tau} \right),$$

where  $\gamma$  is a certain constant factor,  $\vartheta[\cdot]$  is the Riemann  $\theta$  function,

$$\vartheta \left[ \begin{smallmatrix} a \\ b \end{smallmatrix} \right] (z; \boldsymbol{\tau}) := \sum_{n \in \mathbb{Z}^g} \exp \left[ 2\pi \sqrt{-1} \left\{ \frac{1}{2} {}^t(n+a) \boldsymbol{\tau} (n+a) + {}^t(n+a)(z+b) \right\} \right],$$

with  $\boldsymbol{\tau} := \boldsymbol{\omega}'^{-1} \boldsymbol{\omega}''$  for  $g$ -dimensional vectors  $a$  and  $b$ , and

$$\delta' := {}^t \begin{bmatrix} \frac{g}{2} & \frac{g-1}{2} & \dots & \frac{1}{2} \end{bmatrix}, \quad \delta'' := {}^t \begin{bmatrix} \frac{1}{2} & \dots & \frac{1}{2} \end{bmatrix}.$$

**Definition 2.3.** (1) Hyperelliptic  $al$  function is defined by [[Ba2] p.340, [W1]],

$$(2.3) \quad al_r(u) = \gamma_r \sqrt{F(b_r)},$$

where  $\gamma_r := \sqrt{-1/P'(b_r)}$  and

$$(2.4) \quad F(x) := (x - x_1) \cdots (x - x_g).$$

(2) Hyperelliptic  $\zeta_\nu$  function is defined by,

$$(2.5) \quad \zeta_\mu = \frac{\partial}{\partial u_\mu} \log \sigma(u).$$

(3) Hyperelliptic  $\wp_{\mu\nu}$  function is defined by,

$$\wp_{\mu\nu} = -\frac{\partial^2}{\partial u_\mu \partial u_\nu} \log \sigma(u).$$

(4) The power symmetric function  $q$  is defined by

$$(2.6) \quad q_n := \sum_{i=1}^g x_i^n(u), \quad q_{n,\mu} := \frac{\partial}{\partial u_\mu} q_n.$$

On the choice of  $\gamma_r$ , we will employ the convention of Baker [Ba2] instead of original one [W1].

**Proposition 2.1.** (1) *Introducing the half-period  $\omega_r := \int_{\infty}^{b_r} du^{(a)}$ , we have the relation* [[Ba2], p.340],

$$(2.7) \quad \text{al}_r(u) = \gamma_r'' \frac{\exp(-{}^t u \boldsymbol{\eta}' \boldsymbol{\omega}'^{-1} \omega_r) \sigma(u + \omega_r)}{\sigma(u)},$$

where  $\gamma_r''$  is a certain constant.

(2) *The hyperelliptic  $\wp_{gi}$  function is given as an elementary symmetric function,*

$$F(x) = x^g - \sum_{i=1}^g \wp_{g,i} x^{g-i}.$$

i.e.,

$$(2.8) \quad \wp_{g\nu} = (-1)^i e_{\mu-1}(u),$$

where  $e_{\mu}(u)$  is the  $\mu$ -th elementary symmetric function of  $x_i$ 's.

### 3. RELATIONS IN A LOOP SOLITON

As mentioned in Introduction, this section gives relations in a quantized elastica following the previous results. Before we will show our new results, we review the previous results in [Ma3] as follows.

**Proposition 3.1.** *Let the configuration of the  $x$ -components  $(x_1, \dots, x_g)$  of the affine coordinates of the hyperelliptic curves  $\text{Sym}^g(C_g)$  and the coefficients  $\lambda$ 's of each  $C_g$  satisfy,*

$$(3.1) \quad |F(b_r)| = 1, \quad \text{and}, \quad u_g \in \mathbb{R}.$$

*For such  $(x_1, y_1), \dots, (x_g, y_g)$ , we have  $\mathbf{u} := \mathbf{u}((x_1, y_1), \dots, (x_g, y_g))$  and following results.*

(1) *By setting  $s \equiv u_g$  and  $t \equiv u_{g-1} + (\lambda_{2g-1} + b_r)u_g$ ,*

$$\partial_{u_g} Z^{(r)} := F(b_r), \quad \text{or} \quad |\partial_{u_g} Z^{(r)}| = 1,$$

*completely characterizes the loop soliton.*

(2) *The shape of loop soliton is given by,*

$$Z^{(r)} = \frac{1}{r_0} \left( b_r^g u_g + \sum_{i=1}^g b_r^{i-1} \zeta_i \right).$$

(3) *The Schwarz derivative of  $Z$  with respect to  $u_g$*

$$(3.2) \quad \{Z^{(r)}, u_g\}_{\text{SD}} = 4\wp_{gg} + 2\lambda_{2g} + 2b_r.$$

Here we should give remarks on the previous results.

- Remark 3.1.** (1) Though we did not notice neither mention in [Ma3, Ma6], we must say that parts of the results in [Ma3, Ma6] had been already obtained in [P] using Riemann  $\theta$  functions.
- (2) In order to satisfy (3.1) for real parts in the Jacobian, we must constraint the coefficients in (2.1) of the curve. However we could not find such conditions in this stage like genus one case in [Mu1]. Thus we plan to consider these conditions.

From the previous results, we automatically have following corollary.

**Corollary 3.1.** *A loop soliton  $Z^{(r)}(u)$  satisfies following relations,*

$$(1)$$

$$(3.3) \quad \partial_{u_g} Z^{(r)}(u + 2\omega'_i) \equiv \partial_{u_g} Z^{(r)}(u), \quad \partial_{u_g} Z^{(r)}(u + 2\omega''_i) \equiv \partial_{u_g} Z^{(r)}(u).$$

$$(2) \text{ When we regard } Z^{(r)} \text{ as a function of } (x_i - b_r)_{i=1, \dots, g},$$

$$(3.4) \quad \overline{\partial_{u_g} Z^{(r)}(x_1 - b_r, \dots, x_g - b_r)} \equiv \partial_{u_g} Z^{(r)} \left( \frac{1}{x_1 - b_r}, \dots, \frac{1}{x_g - b_r} \right).$$

Followings are our main results in this article.

**Theorem 3.1.** *A loop soliton  $Z^{(r)}(u)$  satisfies following relations,*

$$(1)$$

$$\partial_{u_g} Z^{(r)}(u) = b_r^g \exp \left( - \sum_{n=1}^{\infty} \frac{q_n}{n} b_r^{-n} \right)$$

$$(2)$$

$$(3.5) \quad \{Z^{(r)}(u + \omega_r), u_g\}_{\text{SD}} + \{Z^{(r)}(u), u_g\}_{\text{SD}} = - \sum_{n,m=1}^{\infty} \frac{q_{n,g} q_{m,g}}{nm} b_r^{-n-m}.$$

$$(3)$$

$$(3.6) \quad \frac{1}{2} [\{Z^{(r)}(u + \omega_r), u_g\}_{\text{SD}} - \{Z^{(r)}(u), u_g\}_{\text{SD}}] = -\partial_{u_g}^2 \log (\partial_{u_g} Z^{(r)}(u)).$$

*Proof.* The first formula is obvious from the relation between  $F(b_r)$  and the power symmetric functions  $q_n$ . Due to (2.7), we have

$$-\partial_{u_g}^2 \log(F(b_r)) = -2\wp_{gg}(u + w_r) + 2\wp_{gg}(u).$$

(3.2) leads us the third formula (3.6).

From (3.27) in [Ma4], which is essentially the Miura transformation, we have

$$-\partial_{u_g}^2 \log(F(b_r)) = 4\wp_{gg}(u) + 2\lambda_{2g} + 2b_r + \frac{1}{2} (\partial_{u_g} \log(F(b_r)))^2.$$

As mentioned in (3.8) of [Ma4],

$$\frac{\partial}{\partial u_g} = \sum_{i=1}^g \frac{2y_i}{F'(x_i)} \frac{\partial}{\partial x_i},$$

we have

$$\partial_{u_g} \log(F(b_r)) = \sum_{n=1}^{\infty} \frac{q_{n,g}}{n} b_r^{-n}.$$

These constitute the second formula (3.5).  $\square$

**Remark 3.2.** (1) (3.6) is the generalization of (4.6) in [Ma3], which is the same formula of genus one, to a general genus  $g$ .

(2)  $F(b_r)$  can be regarded as a generation function of the elementary symmetric functions and thus behind our theorem, the Newton formula plays important roles.

(3) It is noted that  $F(x)$  and  $\partial_{u_g} F(x)$  appeared in the book of Mumford [Mu0] as  $U(x)$  and  $V(x)$  in his triplet representation  $(U, V, W)$  of functions of hyperelliptic curves.

(4) (3.3) can be a stronger relation for a closed loop soliton, *i.e.*,

$$(3.7) \quad Z^{(r)}(u_1, \dots, u_{g-1}, u_g + 1) = Z^{(r)}(u_1, \dots, u_{g-1}, u_g).$$

In this case, we have Fourier expansion as shown in next section.

(5) On a loop soliton and geometry of MKdV equations, readers should consult the reference [P], which gives several mathematical open problems and results in issues related to our quantized elastica problem.

(6) As remarked in ([P], 4.3), our system is closely related to the formula 27 in p.19 of [F], which is of the Schwarz derivative and prime form. The al-functions are solutions of the Dirac equation in [Ma3], which is the spinor representation of the Frenet-Serret equation. We should connect them in future.

#### 4. WINDING LOOPS

Due to the (3.7), we have Fourier expansions of  $Z^{(r)}$  of a closed loop soliton or a quantized elastica, *i.e.*, using functions  $a_n$  of  $(u_1, u_1, \dots, u_{g-1})$  and real parameter  $s$ ,

$$Z^{(r)}(u) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} a_n e^{2\pi\sqrt{-1}ns}, \quad \frac{1}{\sqrt{-1}} \partial_s Z^{(r)}(u) = \sum_{n=-\infty}^{\infty} \sqrt{2\pi n} a_n e^{2\pi\sqrt{-1}ns}.$$

In this sense, we will regard  $Z^{(r)}(u)$  as a function of  $s$  with parameters  $u^\# := (u_1, u_1, \dots, u_{g-1})$  and refer it by  $Z^{(r)}(s) := Z^{(r)}(u^\#; s)$ . Then we have the following proposition.

**Proposition 4.1.** (1) *The euclidean move is represented by a choice of  $a_0$  and global constant factor  $c$  of  $Z^{(r)}$ .*

(2) In terms of the real parameter  $s$ , there exists a complex number  $c$ ,

$$\overline{Z^{(r)}(u^\#; s) - a_0} = c(Z^{(r)}(u^\#; -s) - a_0), \quad \overline{a_n} = ca_n, \quad \text{for } n \neq 0.$$

(3) By choosing  $a_0 = 0$  and  $c = 1$ , the reality condition  $|\partial_s Z(u^\#; s)| = 1$  is expressed by

$$2\pi \sum_{m=-\infty}^{\infty} n(n+m)a_m a_{n+m} = \delta_{n,0}.$$

(4) For  $a_0 = 0$  and  $c = 1$ , the fourier coefficients of the curvature of  $Z^{(r)}(-s)$  can be expressed by the bilinear form of  $a_n$ 's,

$$\frac{1}{\sqrt{-1}} \partial_s \log \partial_s Z^{(r)}(u^\#; s) = - \sum_{n=-\infty}^{\infty} \left( 4\pi^2 \sum_{m=-\infty}^{\infty} (n+m)^2 (m) a_m a_{n+m} \right) e^{2\pi\sqrt{-1}ns}.$$

*Proof.* The first relation is obvious. The complex conjugate determines the orientation of the complex plane while that of a loop is by the orientation of the arclength parameter  $s$ . Thus the second relation is justified. Noting  $\overline{\partial_s Z^{(r)}} = 1/\partial_s Z^{(r)}$ , direct computations give the third and fourth relations.  $\square$

As a loop soliton and an element in  $\mathcal{M}_{\text{elas}}^{\mathbb{C}}$  defined as a loop modulo euclidean moves, we should regard  $Z^{(r)}$  as a vector in the complex plane. It implies that an addition of  $Z^{(r)}$ 's with complex coefficients has mathematical meanings.

Further as mentioned in [Ma2], there are winding solutions in our moduli space  $\mathcal{M}_{\text{elas}}^{\mathbb{C}}$ . Hence we will define a winding loop soliton for a loop soliton  $Z^{(r)}(u^\#; s)$ ,

$$(4.1) \quad Z^{(r,n)}(u^\#; s) := \frac{1}{n} Z^{(r)}(u^\#; ns).$$

The winding loop solitons has following properties, which are not difficult to be proved.

**Proposition 4.2.** *For a natural number  $n$  and a prime number  $p$ , we have following relations,*

(1)

$$(4.2) \quad \mathcal{E}[Z^{(r,n)}] = n^2 \mathcal{E}[Z^{(r)}].$$

(2)

$$pZ^{(r,pn)}(u^\#; s) = Z^{(r,n)}(u^\#; ps).$$

(3)

$$(4.3) \quad \left( Z^{(r,pn)} \left( \frac{s}{p} \right) + Z^{(r,pn)} \left( \frac{s+1}{p} \right) + \cdots + Z^{(r,pn)} \left( \frac{s+p-1}{p} \right) \right) = Z^{(r,n)}(s).$$



**Remark 4.1.** (1) The relation (4.3) remind of the action of Hecke for modular function of vanishing weight and for a prime number  $p$  [S],

$$(4.4) \quad pT_p(f(z)) = f(pz) + \left( f\left(\frac{z}{p}\right) + f\left(\frac{z+1}{p}\right) + \cdots + f\left(\frac{z+p-1}{p}\right) \right).$$

- (2) We should note the partition function (1.3). Even though (1.3) could not computed in this stage, we can compute its part,  $\mathcal{Z}^{(a)}[\beta]$  ( $a = 1, 2$ ). We know the closed loop soliton solutions of genera zero and one explicitly, which is given by disjoint types, *i.e.*, a circle and an eight-figure shape [Ma1]. Considering contributions of winding loop soliton, for  $a = 0, 1$ , we obtain,

$$\mathcal{Z}^{(a)}[\beta] = \sum_{n=1}^{\infty} e^{-\beta n^2 E_a} = \frac{1}{2} (\theta(\sqrt{-1}\beta E_a/\pi) - 1),$$

where  $E_0$  and  $E_1$  are the energies of genera zero and one and  $\theta(z)$  is the elliptic theta function,  $\theta(z) := \sum_{n=-\infty}^{\infty} e^{\sqrt{-1}\pi z n^2}$ . Due to properties of the elliptic theta function and Poisson sum formula,

$$\mathcal{Z}^{(a)}[\beta] = \sqrt{\frac{1}{E_a\beta}} \sum_{n=1}^{\infty} e^{-n^2/E_a\beta} + \frac{1}{2} \left( \frac{1}{\sqrt{E_a\beta}} - 1 \right).$$

As  $\mathcal{Z}^{(a)}[\beta + 2\pi\sqrt{-1}/E_a] = \mathcal{Z}^{(a)}[\beta]$ , we regard that  $\mathcal{Z}^{(a)}[\beta]$  has modular properties.

When we approximate  $\mathcal{Z}[\beta]$  by  $\mathcal{Z}^{(a)}[\beta]$  or  $\mathcal{Z}^{(0,1)}[\beta] := \sum_{a=0}^1 \mathcal{Z}^{(a)}[\beta]$ , we might encounter a critical phenomena from the viewpoint of statistical physics due to the modular properties.

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